MATH2050C Final Examination

Answer all five questions. You should justify your answer.

1. (a) (10 marks) Use ε-δ definition to determine the following limit

$$\lim_{x\to 1} \frac{x^2-1}{3x^2-2x-1} .$$

   (b) (5 marks) Determine the limit in (a) using Limit Theorem.

Solution (a) For $x \neq 1$, $\frac{x^2-1}{3x^2-2x-1} = \frac{x+1}{3x+1}$. We have

$$\left|\frac{x+1}{3x+1} - \frac{1}{2}\right| = \left|\frac{x-1}{2(3x+1)}\right| .$$

For $x, |x-1| < 1/2, x > 1 - 1/2 = 1/2$. Therefore, $1/2(3x+1) < 1/2(3/2+1) = 1/5$ so

$$\left|\frac{x+1}{3x+1} - \frac{1}{2}\right| = \left|\frac{x-1}{2(3x+1)}\right| < \frac{|x-1|}{5} .$$

For $\varepsilon > 0$, let $\delta = \min\{1/2, 5\varepsilon\}$, then

$$\left|\frac{x^2-1}{3x^2-2x-1} - \frac{1}{2}\right| = \left|\frac{x+1}{3x+1} - \frac{1}{2}\right| < \frac{|x-1|}{5} < \frac{5\varepsilon}{5} = \varepsilon.$$

We conclude that

$$\lim_{x\to 1} \frac{x^2-1}{3x^2-2x-1} = \frac{1}{2} .$$

(b) Clearly, $\lim_{x\to 1}(x+1) = 2$ and $\lim_{x\to 1}(3x+1) = 4$. By Limit Theorem (or Quotient Rule),

$$\lim_{x\to 1} \frac{x^2-1}{3x^2-2x-1} = \frac{\lim_{x\to 1}(x+1)}{\lim_{x\to 1}(3x+1)} = \frac{1}{2} .$$

2. Determine the largest domains on which these functions are defined and study their continuity:

   (a) (10 marks) $\sqrt{\dfrac{x-1}{x+2}}$.

   (b) (10 marks) $g \circ f(x)$ where $f(x) = x^2$ and $g(z) = 1 - z, z \geq 0$ and $g(z) = 0, z < 0$.

Solution (a) The largest domain of definition is $(-\infty, -2) \cup [1, \infty)$ and the function is continuous there.

(b) This function is continuous on $(-\infty, \infty)$.

3. (a) (5 marks) Let $\{x_n\}$ be a sequence of positive numbers converging to x. Use the ε-n_0 definition to show that x is non-negative.

(b) (10 marks) Let f be a continuous function on $[a,b]$ satisfying $f(a)<0$ and $f(b)>0$. Use the bisection method to show that there is some $c\in(a,b)$ satisfying $f(c)=0$.

(c) (5 marks) Prove that the function $g(x)=x^2-64x\sin x+18\cos x$ has at least two roots.

Solution (a) Suppose on the contrary x is negative. Taking $\varepsilon=|x|/2$, there is some n_0 such that $|x_n-x|<|x|/2$ for all $n\geq n_0$. But then $x_n-x<|x|/2$ which implies $x_n<x+|x|/2<0$, contradiction holds. (Other correct solutions are allowed.)

Solution (b) See textbook.

Solution (c) Observe that $g(\pi)=\pi^2-18<0$ and

$$g(x)=x^2\left(1-\frac{64\sin x}{x}+\frac{18\cos x}{x^2}\right)\to\infty\ ,$$

respectively as $x\to\pm\infty$. We can fix a large $M>0$ such that $g(\pm M)>0$. Applying the Location of Root Theorem to g and $[-M,\pi]$ we conclude g has a root in $(-M,\pi)$. By the same reasoning, it also has a root in (π,M). Therefore, g has at least two (real) roots.

4. (a) (10 marks) Show that $\sin 1/x$ is not uniformly continuous on $(0,1]$.

   (b) (10 marks) Let F be a continuous function on $[-1,1]$. Suppose the function $F(\sin 1/x)$ is uniformly continuous on $(0,2\pi]$. Show that F must be a constant function.

Solution (a) If $f(x)\equiv\sin 1/x$ is uniformly continuous on $(0,1]$, then for $\varepsilon=1$, there is some $\delta>0$ such that $|f(x)-f(y)|<1$ whenever $|x-y|<\delta$. However, taking $x_n=1/2n\pi$ and $y_n=1/(2n\pi+1/2)$, we have $|x_n-y_n|<\delta$ for large n and yet $|f(x_n)-f(y_n)|=1$, contradiction holds. Hence f is not uniformly continuous on $(0,1]$.

(b) As $F(\sin 1/x)$ is uniformly continuous on $(0,2\pi]$, it extends as a continuous function on $[0,2\pi]$. Pick any two numbers $a,b\in[-1,1]$. We can find $\{x_n\},x_n\to0$, such that $\sin 1/x_n\to a$, and $\{y_n\},y_n\to0$ such that $\sin 1/y_n\to b$ as $n\to\infty$. Since $x_n,y_n\to0$, by the continuity of $F(\sin 1/x)$ at $x=0$, $F(a)=\lim_{n\to\infty}F(\sin 1/x_n)=\lim_{n\to\infty}F(\sin 1/y_n)=F(b)$. We conclude that F is a constant function.

5. (a) (10 marks) Show that for each $x\geq0$,

$$E(x)\equiv\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n\ ,$$

   exists.

   (b) (10 marks) Show that the limit in (a) exists for all real numbers x and $E(x)E(-x)=1$ holds.

   (c) (5 marks) Show that

$$\lim_{t\to\infty}\left(1+\frac{x}{t}\right)^t\ ,\ x\in\mathbb{R}\ ,$$

   exists and is equal to $E(x)$.

Solution (a) See textbook or my notes.

(b) See Exercise.

(c) Assume $x \geq 0$. (The negative case can be treated by a similar way.) Let $\{t_n\}$ be any sequence $t_n \to \infty$, we can find $m \to \infty$ such that $m \leq t_n < m+1$. Then $1 + x/(m+1) \leq 1 + x/t \leq 1 + x/m$. It follows that

$$\left(1 + \frac{x}{m+1}\right)^{t_n} \leq \left(1 + \frac{x}{t_n}\right)^{t_n} \leq \left(1 + \frac{x}{m}\right)^{t_n} ,$$

so

$$\left(1 + \frac{x}{m+1}\right)^{m+1} \left(1 + \frac{x}{m+1}\right)^{-1} \leq \left(1 + \frac{x}{t_n}\right)^{t_n} \leq \left(1 + \frac{x}{m}\right)^{m} \left(1 + \frac{x}{m}\right) .$$

By squeezing,

$$E(x) \leq \lim_{t_n \to \infty} \left(1 + \frac{x}{t_n}\right)^{t_n} \leq E(x) .$$

By the Sequential Criterion for limit,

$$\left(1 + \frac{x}{t}\right)^{t} \to E(x),$$

as $t \to \infty$ for $x \geq 0$.